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I. THE PROBLEM

The time evolution of a coherently driven two-state quantum system is described by the two coupled ordinary differential equations for the probability amplitudes $C_1(t)$ and $C_2(t)$ of states ψ_1 and ψ_2 :

$$i \frac{d}{dt} C_1(t) = \frac{1}{2} \Omega(t) C_2(t), \quad (1a)$$

$$i \frac{d}{dt} C_2(t) = \frac{1}{2} \Omega(t) C_1(t) + \Delta(t) C_2(t). \quad (1b)$$

where $\Omega(t)$ is the coupling (assumed real) between the two states and $\Delta(t)$ is the frequency detuning.

In the Rosen-Zener (RZ) model [1–5], the coupling and the detuning are given by

$$\Omega(t) = \Omega_0 \operatorname{sech}(t/T), \quad \Delta(t) = \Delta_0. \quad (2)$$

The real constants Ω_0 and Δ_0 have the dimension of frequency and will be assumed positive without loss of generality. The positive parameter T is the characteristic pulse width. The pulse area is

$$\mathcal{A} = \int_{-\infty}^{+\infty} \Omega(t) dt = \pi \Omega_0 T. \quad (3)$$

The probability amplitudes $C_1(t_f)$ and $C_2(t_f)$ at the final time of interaction t_f are connected to their values $C_1(t_i)$ and $C_2(t_i)$ at the initial time t_i by the evolution matrix $\mathbf{U}(t_f, t_i)$

$$\mathbf{C}(t_f) = \mathbf{U}(t_f, t_i) \mathbf{C}(t_i), \quad (4)$$

where $\mathbf{C}(t) = [C_1(t), C_2(t)]^T$. Because the interaction is coherent, the probability is conserved and $\mathbf{U}(t_f, t_i)$ is a unitary matrix, $\mathbf{U}^{-1}(t_f, t_i) = \mathbf{U}^\dagger(t_f, t_i)$.

II. THE EXACT SOLUTION

The first step is to decouple Eqs. (1) by differentiating the equation for $\dot{C}_1$ (with the overdot denoting a time derivative) and replacing C_2 and $\dot{C}_2$, found from Eqs. (1); this gives

$$\ddot{C}_1 - \left(\frac{\dot{\Omega}}{\Omega} - i\Delta \right) \dot{C}_1 + \frac{1}{4} \Omega^2 C_1 = 0. \quad (5)$$

The next step is to change the independent variable from t to

$$z(t) = \frac{1}{2} [\tanh(t/T) + 1]. \quad (6)$$

Notice that $z(-\infty) = 0$ and $z(+\infty) = 1$. Taking into account that

$$\frac{d}{dt} = \dot{z} \frac{d}{dz}, \quad \frac{d^2}{dt^2} = \ddot{z} \frac{d}{dz} + \dot{z}^2 \frac{d^2}{dz^2}, \quad (7a)$$

$$\dot{z} = \frac{1}{2T \cosh^2(t/T)} = \frac{1}{2T} [1 - \tanh^2(t/T)] = \frac{2z(1-z)}{T}, \quad (7b)$$

$$\ddot{z} = \frac{4z(1-z)(1-2z)}{T^2}, \quad (7c)$$

$$\Omega^2 = 4\Omega_0^2 z(1-z), \quad \frac{\dot{\Omega}}{\Omega} = -\frac{\sinh(t/T)}{T \cosh(t/T)} = \frac{1-2z}{T}, \quad (7d)$$

Eq. (5) is transformed into

$$z(1-z)\frac{d^2C_1}{dz^2} + \left[\frac{1}{2} - z + \frac{1}{2}i\Delta_0 T\right]\frac{dC_1}{dz} + \frac{1}{4}\Omega_0^2 T^2 C_1 = 0. \quad (8)$$

This equation has the same form as the Gauss hypergeometric equation (A1) [7,6,8], upon the identification

$$\lambda = \frac{1}{2}\Omega_0 T, \quad \mu = -\frac{1}{2}\Omega_0 T, \quad \nu = \frac{1}{2} + \frac{1}{2}i\Delta_0 T. \quad (9)$$

The solution of Eq. (8) can be expressed in terms of the Gauss hypergeometric function [7,8] as

$$C_1(z) = AF(\lambda, \mu; \nu; z) + Bz^{1-\nu}F(\lambda + 1 - \nu, \mu + 1 - \nu; 2 - \nu; z). \quad (10)$$

From here and using Eq. (1a) and the derivative properties (A2), it can be found that

$$C_2(z) = \frac{2i\sqrt{z(1-z)}}{\Omega_0 T} \left[A \frac{\lambda\mu}{\nu} F(\lambda + 1, \mu + 1; \nu + 1; z) + B(1-\nu)z^{-\nu}F(\lambda + 1 - \nu, \mu + 1 - \nu; 1 - \nu; z) \right]. \quad (11)$$

The integration constants A and B are to be determined from the initial conditions. By taking into account that $z(t \rightarrow -\infty) = 0$ and Eq. (A4a) we find

$$A = C_1(0), \quad B = \frac{\Omega_0 T z^{\frac{1}{2}i\Delta_0 T}}{2i(1-\nu)} C_2(0). \quad (12)$$

Hence,

$$C_1(z) = F(\lambda, \mu; \nu; z)C_1(0) + F(\lambda + 1 - \nu, \mu + 1 - \nu; 2 - \nu; z) \frac{\Omega_0 T}{2i(1-\nu)} C_2(0), \quad (13)$$

and, recalling that $z(-\infty) = 0$ and $z(+\infty) = 1$, we find

$$U_{11}(+\infty, -\infty) = F(\lambda, \mu; \nu; 1) = \frac{\Gamma(\nu)\Gamma(\nu - \lambda - \mu)}{\Gamma(\nu - \lambda)\Gamma(\nu - \mu)} = \frac{\Gamma^2(\frac{1}{2} + \frac{1}{2}i\Delta_0 T)}{\Gamma(\frac{1}{2} + \frac{1}{2}i\Delta_0 T - \frac{1}{2}\Omega_0 T)\Gamma(\frac{1}{2} + \frac{1}{2}i\Delta_0 T + \frac{1}{2}\Omega_0 T)}, \quad (14a)$$

$$\begin{aligned} U_{12}(+\infty, -\infty) &= F(\lambda + 1 - \nu, \mu + 1 - \nu; 2 - \nu; 1) \frac{\Omega_0 T}{2i(1-\nu)} = -i \frac{\lambda}{1-\nu} \frac{\Gamma(2-\nu)\Gamma(-\lambda-\mu+\nu)}{\Gamma(1-\lambda)\Gamma(1-\mu)} \\ &= -i \frac{\Gamma(1-\nu)\Gamma(\nu)}{\Gamma(1-\frac{1}{2}\Omega_0 T)\Gamma(\frac{1}{2}\Omega_0 T)} = -i \frac{\sin(\frac{1}{2}\pi\Omega_0 T)}{\cosh(\frac{1}{2}\pi\Delta_0 T)}, \end{aligned} \quad (14b)$$

where we have used Eq. (A4b) and the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$.

The populations are $P_1(+\infty) = |U_{11}(+\infty, -\infty)|^2$ and $P_2(+\infty) = |U_{12}(+\infty, -\infty)|^2$, or explicitly,

$$\begin{aligned} P_1(+\infty) &= \frac{[\Gamma(\frac{1}{2} + \frac{1}{2}i\Delta_0 T)\Gamma(\frac{1}{2} - \frac{1}{2}i\Delta_0 T)]^2}{\Gamma(\frac{1}{2} + \frac{1}{2}i\Delta_0 T - \frac{1}{2}\Omega_0 T)\Gamma(\frac{1}{2} - \frac{1}{2}i\Delta_0 T - \frac{1}{2}\Omega_0 T)\Gamma(\frac{1}{2} + \frac{1}{2}i\Delta_0 T + \frac{1}{2}\Omega_0 T)\Gamma(\frac{1}{2} - \frac{1}{2}i\Delta_0 T + \frac{1}{2}\Omega_0 T)} \\ &= \frac{\cos[\frac{1}{2}\pi(\Omega_0 + i\Delta_0)T] \cos[\frac{1}{2}\pi(\Omega_0 - i\Delta_0)T]}{\cosh^2(\frac{1}{2}\pi\Delta_0 T)} = \frac{\cos(\pi\Omega_0 T) + \cosh(\pi\Delta_0 T)}{2 \cosh^2(\frac{1}{2}\pi\Delta_0 T)} = 1 - \frac{\sin^2(\frac{1}{2}\pi\Omega_0 T)}{\cosh^2(\frac{1}{2}\pi\Delta_0 T)} \end{aligned} \quad (15a)$$

$$P_2(+\infty) = \frac{\sin^2(\frac{1}{2}\pi\Omega_0 T)}{\cosh^2(\frac{1}{2}\pi\Delta_0 T)}. \quad (15b)$$

where we have used the modified reflection formula $\Gamma(\frac{1}{2} + z)\Gamma(\frac{1}{2} - z) = \pi/\cos(\pi z)$.

III. PROBLEMS

PROBLEM 1. Derive the evolution matrix and the transition probability for the *Allen-Eberly-Hioe model*,

$$\Omega(t) = \Omega_0 \text{sech}(t/T), \quad \Delta(t) = B \tanh(t/T). \quad (16)$$

Answer.

$$U_{11}(+\infty, -\infty) = \frac{\Gamma(\nu)\Gamma(\nu - \lambda - \mu)}{\Gamma(\nu - \lambda)\Gamma(\nu - \mu)}, \quad (17)$$

$$\lambda = \frac{1}{2}\sqrt{\Omega_0^2 - B^2}T - \frac{1}{2}iBT, \quad \mu = -\frac{1}{2}\sqrt{\Omega_0^2 - B^2}T - \frac{1}{2}iBT, \quad \nu = \frac{1}{2} - \frac{1}{2}iBT, \quad (18)$$

$$P_2 = 1 - \frac{\cos^2\left(\frac{1}{2}\pi T\sqrt{\Omega_0^2 - B^2}\right)}{\cosh^2\left(\frac{1}{2}\pi BT\right)}. \quad (19)$$

PROBLEM 2. Derive the evolution matrix and the transition probability for the *Bambini-Berman model*,

$$\Omega(t) = \Omega_0 \operatorname{sech}(t/T), \quad \Delta(t) = B [\tanh(t/T) + 1]. \quad (20)$$

Answer.

$$U_{11}(+\infty, -\infty) = \frac{\Gamma(\nu)\Gamma(\nu - \lambda - \mu)}{\Gamma(\nu - \lambda)\Gamma(\nu - \mu)}, \quad (21)$$

$$\lambda = \frac{1}{2}\sqrt{\Omega_0^2 - B^2}T - \frac{1}{2}iBT, \quad \mu = -\frac{1}{2}\sqrt{\Omega_0^2 - B^2}T - \frac{1}{2}iBT, \quad \nu = \frac{1}{2}, \quad (22)$$

$$P_2 = \frac{1}{2} - \frac{1}{2} \frac{\cos\left(\pi T\sqrt{\Omega_0^2 - B^2}\right)}{\cosh(\pi BT)}. \quad (23)$$

PROBLEM 3. Derive the evolution matrix and the transition probability for the *Demkov-Kunike model*,

$$\Omega(t) = \Omega_0 \operatorname{sech}(t/T), \quad \Delta(t) = \Delta_0 + B \tanh(t/T). \quad (24)$$

Answer.

$$U_{11}(+\infty, -\infty) = \frac{\Gamma(\nu)\Gamma(\nu - \lambda - \mu)}{\Gamma(\nu - \lambda)\Gamma(\nu - \mu)}, \quad (25)$$

$$\lambda = \frac{1}{2}\sqrt{\Omega_0^2 - B^2}T - \frac{1}{2}iBT, \quad \mu = -\frac{1}{2}\sqrt{\Omega_0^2 - B^2}T - \frac{1}{2}iBT, \quad \nu = \frac{1}{2} + \frac{1}{2}i(\Delta_0 - B)T, \quad (26)$$

$$P_2 = \frac{\cosh(\pi BT) - \cos\left(\pi T\sqrt{\Omega_0^2 - B^2}\right)}{\cosh(\pi \Delta_0 T) + \cosh(\pi BT)}. \quad (27)$$

APPENDIX A: RELEVANT PROPERTIES OF THE GAUSS HYPERGEOMETRIC FUNCTION

The Gauss hypergeometric equation is [7,6,8],

$$z(1-z)w'' + [\nu - (\lambda + \mu + 1)z]w' - \lambda\mu w = 0. \quad (A1)$$

It possesses two independent solutions, $f(z) = F(\lambda, \mu; \nu; z)$ and $g(z) = z^{1-\nu}F(\lambda + 1 - \nu, \mu + 1 - \nu; 2 - \nu; z)$. The Gauss function has the derivative property

$$\frac{d}{dz}F(a, b; c; z) = \frac{ab}{c}F(a + 1, b + 1; c + 1; z). \quad (A2a)$$

$$\frac{d}{dz}[z^{c-1}F(a, b; c; z)] = (c-1)z^{c-2}F(a, b; c-n; z) \quad (A2b)$$

$$W \{f(z), g(z)\} \equiv f(z) \frac{dg(z)}{dz} - g(z) \frac{df(z)}{dz} = (1-c)z^{-c}(1-z)^{c-a-b-1}. \quad (\text{A3})$$

Special value of the argument

$$F(a, b; c; 0) = 1, \quad (\text{A4a})$$

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \quad (\text{A4b})$$

Power series expansion

$$F(a, b; c; z) = 1 + \frac{ab}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)} \frac{z^3}{3!} + \dots \quad (\text{A5})$$

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